Lecture 8: Surfaces w/ $\kappa = 0$

We will study the minimal models w/ $\kappa = 0$

- $\kappa = 0 \Leftrightarrow P_n = 0$ or 1, $P_n = 1$ for some $n \in \mathbb{N}$.

- X: minimal model w/ $\kappa \geq 0$
  $\Longrightarrow K_X \cdot D \geq 0$, for effective divisor D. In particular, $K_X^2 \geq 0$
  If $K_X . C < 0$, then $C^2 \geq 0$. Thus $|nK_X| = \phi$, or $\kappa = -\infty$

Lemma 1: X minimal surface w/ $\kappa = 0$

Then (1) $K_X^2 = 0$

(2) $\chi(\mathcal{O}_S) \geq 0$ In particular, $q \leq 2$

(3) $P_n = P_m = 1$, then $P_d = 1$, $d = \gcd(m, n)$.

pf: (1) Notice that $K_X^2 \geq 0$.

$$h^0(nK_X) + \underset{\substack{\| \\ h^0(-(n-1)K_X) \\ \| \\ 0}}{h^2(nK_X)} \geq \chi(\mathcal{O}_X) + \underset{\substack{\| \\ \frac{1}{2}K_X^2 \cdot n^2 + O(n)}}{\frac{1}{2} nK_X \cdot (nK_X - K_X)}$$

unless $K_X^2 \leq 0$

(2) Noether's formula

$$\chi(\mathcal{O}_X) = \frac{1}{12}(\cancel{K_X^2} + \chi_{top}(X)) = \frac{1}{12}(2 - 2b_1 + b_2)$$

with $K_X^2 = 0$.

$$12(\underset{1}{h^0(\mathcal{O}_X)} - \underset{q}{h^1(\mathcal{O}_X)} + \underset{\substack{\| \\ h^0(2K_X) \\ \geq \\ 1}}{h^2(\mathcal{O}_X)}) = 2 - \underset{4q}{2b_1} + b_2$$

$$8\chi(\mathcal{O}_X) \geq -6 + b_2 \quad \therefore \quad \underset{\substack{\cap \\ \mathbb{Z}}}{\chi(\mathcal{O}_X)} \geq \frac{-6}{8} + \frac{1}{8}b_2 > -1$$

(3) $m = m'd$, $n = n'd$, $(m', n') = 1$

$D \in |mK_X|$, $E \in |nK_X|$

then $n'D, m'E \in |\frac{mn}{d} K_X|$ of dimension 0.

$\therefore$ $n'D = m'E$ as Weil divisors

$\therefore$ $m' | D$ or $D = m'D'$, $D' \in |dK_X| \neq \phi$ $\square$

Theorem 1. X = minimal surface w/ $\kappa = 0$. Then it is one of below:

(1) $P_g = 0$, $q = 0$; then $2K_X \equiv 0$ Enrique surface.

(2) $P_g = 0$, $q = 1$; then X is a bi-elliptic surface.

(3) $P_g = 1$, $q = 0$; then $K \equiv 0$, K3 surface

(4) $P_g = 1$, $q = 2$; then X is an abelian surface.

pf: $\kappa = 0 \Rightarrow P_g \leq 1$, Lemma 1 (2) $\Rightarrow q \leq 2$

- $P_g = 0$, $q = 0$, then

$$h^0(-2K_X) + h^0(3K_X) \geq \underset{\underset{h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)}{\|}}{\chi(\mathcal{O}_X)} + \frac{1}{2}(-2K_X)\cdot(-3K_X) \geq 1.$$

$$\underbrace{h^0(\mathcal{O}_X)}_{=1} - \underbrace{\cancel{h^1(\mathcal{O}_X)}}_{q=0} \to 0 + \underbrace{\cancel{h^2(\mathcal{O}_X)}}_{P_g=0} \to 0$$

$$\kappa = 0, P_g = 0 \xrightarrow[\text{Castelnuovo}]{} P_2(X) \neq 0 \xrightarrow[\substack{\text{Lemma 1 (3)} \\ P_g = 0}]{} P_3(X) = 0$$

$$\Longrightarrow h^0(-2K_X) \neq 0 \quad \text{or} \quad 2K_X = 0.$$

- $P_g = 0$, $q \geq 1$, $\kappa = 0 \xrightarrow[\text{classification}]{} $ bi-elliptic surface.

- $P_g = 1,\ q = 0,$

$$\underset{\overset{\shortparallel}{1}}{h^0(2K_X)} + h^0(-K_X) \geq \chi(\mathcal{O}_X) + \tfrac{1}{2} 2K_X . K_X = 2$$

$$\chi(\mathcal{O}_X) = \underset{\overset{\shortparallel}{1}}{h^0(\mathcal{O}_X)} - \underset{\overset{\shortparallel}{0}}{h^1(\mathcal{O}_X)} + \underset{\overset{\shortparallel}{1}}{h^2(\mathcal{O}_X)}$$

$$\therefore \begin{array}{l} h^0(-K_X) \neq 0 \\ h^0(K_X) = P_g = 1 \end{array} \implies K_X = 0$$

- $P_g = 1,\ q = 1$

$-1 \in H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \neq \phi$ is 2-torsion

complex torus

$\epsilon \neq 0$

Pic(X)

ϵ

$$\epsilon \neq 0,\ 2\epsilon = 0 \implies h^0(\epsilon) = h^0(-\epsilon) = 0$$

$$\epsilon = 0$$

R.R $\quad h^0(\epsilon) + h^0(K_X - \epsilon) \geq \chi(\mathcal{O}_X) + \tfrac{1}{2}\epsilon(\epsilon - K_X) = 1$

$\chi(\mathcal{O}_X) = 1 - q + P_g = 1$

$$\implies h^0(K_X - \epsilon) \neq 0,$$

Let $D \in |K_X - \epsilon|,\ E \in |K_X| \implies 2D,\ 2E \in |2K_X|$

$$\underset{P_2(X) = 1}{\implies} 2D = 2E \quad \text{or} \quad D = E$$

$P_g = 1$ + Lemma 1 (3)

$$\implies \epsilon = 0 \quad \rightarrow\!\!\times$$

Thus, no surface w/ $P_g = P_2 = q = 1$.

- $P_g = 1,\ q = 2$. Take $K \in |K_X|$.

(i) Connected Component of K is one of below

* smooth elliptic curve or multiple

* rational curve w/ a node or multiple
* union of smooth rational curves.

If $K_X \neq 0$, write $K = \sum n_i C_i$, $n_i > 0$

$K^2 = 0$, $\underline{K.C_i \geq 0}$ $\Longrightarrow$ $K.C_i = 0$, $\forall i$

$K \geq 0$ + X minimal
$\Rightarrow K_X$ nef

$\Longrightarrow n_i C_i^2 + \sum_{j \neq i} n_j \underbrace{C_i.C_j}_{\geq 0} = 0$

$\Longrightarrow$ ① $C_i^2 = 0$, then $C_i.C_j = 0$ or $C_i \cap C_j = \phi$

$2g-2 = C_i.(C_i+K)$ $\therefore$ $h^1(C_i, \mathcal{O}_{C_i}) = 1$

C_i = smooth elliptic curve
or rational curve w/ a node.

② $C_i^2 < 0$, $2g-2 = C_i.(C_i+K) - C_i^2$

$\therefore$ $C_i^2 = -2$, C_i smooth rat'l curve.

then all irreducible components C_j w/ the same connected component of C_i are smooth rat'l curve, $C_j^2 = -2$.

Now we study the Albanese map $\alpha : X \longrightarrow \alpha(X) \subseteq \mathrm{Alb}(X)$

(ii) $\alpha(X)$ is a curve, then it is smooth of $\underline{\text{genus } 2}$.
$\because q = 2$

If $K \neq 0$, then all components of K are of $g \leq 1$
thus can only fall in the fibres of α.

$K^2 = 0 \Longrightarrow K = \sum_i t_i \alpha^{-1}(y_i)$, $t_i \in \mathbb{Q}$, $y_i \in \alpha(X)$

intersection matrix
of the fibre is negative definite

Then $h^0(X, nK_X) \geq h^0(\alpha(X), nt_iy_i) \longrightarrow \infty$ as $n \gg 0$
if $nt_i \in \mathbb{N}$

Contradicts to $K=0$.

If $K=0$, then generic fibres are smooth curve of genus 1
$F.(F+K_X)=0$

Take an étale cover $\widetilde{B} \longrightarrow B = \alpha(X)$

and set $\widetilde{X} := X \times_B \widetilde{B} \longrightarrow \widetilde{B}$

$$\begin{array}{c} \widetilde{X} \\ \downarrow \text{étale} \\ X \end{array}$$

then $\chi(\mathcal{O}_{\widetilde{X}}) = k\chi(\mathcal{O}_X) = 0 \Longrightarrow q(\widetilde{X}) = 2 \geq h^0(\widetilde{B}, \mathcal{O}_{\widetilde{B}}) = g_{\widetilde{B}} > g_B$

$K_{\widetilde{X}} = 0$ Leray spectral sequence Hurwitz formula

(iii) α is surjective by maximum principle

If $K \neq 0$, let $D =$ a component of K.

$D^2 = 0 \Longrightarrow D$ not contracted by α.

negativity lemma + Stein factorization

$\Longrightarrow D$ has no rational components.

no non-const. map from a rational curve to a complex tori

Otherwise can pull-back nonzero 1-form to $\mathbb{P}^1$

$\underset{(i)}{\Longrightarrow} D = mE$, E smooth elliptic curve

Up to translation, may assume $E \subseteq \alpha(X)$ sub-abelian variety.

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & \alpha(X) \longrightarrow & \alpha(X)/E = B \\ \cup\mathrm{l} & & & \in \\ E & \longrightarrow & & o \end{array}$$

$h^0(nE) \geq h^0(B, \mathcal{O}_B(n \cdot o)) \longrightarrow \infty$ $\therefore h^0(nD) \longrightarrow \infty$ $n \gg 0$
n: suitably divisible

Thus, $h^0(X, nK_X) \longrightarrow \infty$ as $n \gg 0$
n: suitably countable

$K = 0$ $\not\longrightarrow$

Therefore, the only possibility is $K = 0$

$$X \xrightarrow{\alpha} \mathrm{Alb}(X) = A$$

$\omega_1, \omega_2 \in H^0(A, \Omega_A^1)$.

Then $\omega_1 \wedge \omega_2 \in H^0(A, \Omega_A^2) \Longrightarrow \alpha^*(\omega_1 \wedge \omega_2) \in H^0(X, K_X)$

$\Longrightarrow \alpha^*(\omega_1 \wedge \omega_2)$ nowhere vanishing

$K_X \cong \mathcal{O}_X$

$\Longrightarrow \alpha$ is étale everywhere

Thus, X is an étale cover of an abelian surface which is again an abelian surface $\square$